South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.14, No.1 2018, pp. 95-102

ON THE STABILITY OF COMMON NEIGHBOR POLYNOMIAL OF SOME GRAPHS

Shikhi M. and Anil Kumar V.*

Department of Mathematics,
TMG College, Vakkad P.O, Tirur,
Malappuram, Kerala-676502, INDIA
E-mail: shikhianil@gmail.com

*Department of Mathematics,
University of Calicut, Malappuram, Kerala-673635, INDIA
E-mail: anil@uoc.ac.in

(Received: January 04, 2018)

Abstract: Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n - 2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the *common neighbor polynomial* of G. A root of the polynomial $N[G; x]$ is defined as the *common neighbor root* of the graph G. In this paper we study the stability of common neighbor polynomial of some graphs.

Keywords and Phrases: Common neighbor polynomial, common neighbor root.

2010 Mathematics Subject Classification: 05C31, 05C39.

1. Introduction

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n - 2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the *common neighbor polynomial* of G [3]. In [3] the present authors introduced the common neighbor polynomial of graphs and derived the common neighbor polynomial of some well known graphs. Moreover, the common neighbor polynomial of some graph operations were discussed in [4]. A zero of the polynomial $N[G; x]$ is

defined as the *common neighbor root* of the graph G.
A polynomial $f(x_1, \ldots, x_n)$ is said to be *stable* [6] with respect to a region $\Omega \subset \mathcal{C}^n$ if no root of f lies in Ω. Polynomials which are stable with respect to the open left half plane and with respect to the open unit disk are called *Hurwitz polynomial* and *Schur polynomial* respectively. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems [1]. A graph polynomial is worthwhile to study only if it models some stable physical systems. This motivates the authors to study the stability of common neighbor polynomial of some graphs. In this paper we study the stability of common neighbor polynomial with respect to the open left half plane and thus identify the conditions under which the common neighbor polynomial of certain graph classes become a Hurwitz polynomial. All the graph theoretic terminology and notations used in the paper are as in [2].

2. Main Results

Theorem 1. *Zero is a common neighbor root of a graph G if and only if any pair of vertices of G has atleast one common neighbor.*

Proof. Let G be a graph with n vertices. Let zero be a root of $N[G;x]$. Then $N[G;x] = x\, g(x)$, where $g(x)$ is a polynomial of degree one less than that of $N[G;x]$. Then the constant term of $N[G;x]$ is zero and hence the result follows. Conversely, assume that any pair of vertices of G has atleast one common neighbor. Then $|N(G,0)| = 0$. It follows that x is a factor of $N[G;x]$.

Theorem 2. *Zero is the only common neighbor root of a graph G if and only if any two pairs of vertices of G has same number of common neighbors.*

Proof. First assume that zero is the only common neighbor root of a graph G with multiplicity k. Then $N[G;x]$ is of the form Kx^k for some constant K. Since, $\sum_{i=0}^{n-2} |N(G,i)| = \binom{n}{2}$,(see [3]) it follows that $K = \binom{n}{2}$. Thus, $N[G;x] = \binom{n}{2}x^k$ and so all the pairs of vertices of G has k common neighbors. Conversely, assume that all the pairs of vertices of G has exactly k common neighbors. Then the result follows from the fact that $N[G;x] = \binom{n}{2}x^k$.

Example 3. *Zero is the only common neighbor root of the complete graph K_n with multiplicity $n-2$.*

Theorem 4. $(0,\infty)$ *is a zero-free interval of the common neighbor polynomial $N[G;x]$ of any graph G.*

Proof. The result follows from the fact that all the coefficients of $N[G;x]$ are non-negative.

Lemma 5. *(see [3]) For a path P_n with $n \geq 2$ vertices, we have*

$$N[P_n; x] = \ (n-2)x + \binom{n-1}{2} + 1.$$

Theorem 6. *Let P_n be a path with $n > 2$ vertices. Then $N[P_n; x]$ is stable.*

Proof. From lemma 5, $N[P_n; x]$ has a single root $x = \dfrac{-(n-1)(n-2)-2}{2(n-2)}$ which lie in the left half plane since $n > 2$. Hence the result follows.

Lemma 7. *(see [3]) For a cycle C_n with $n \geq 3$ vertices,*

$$N[C_n; x] = \begin{cases} nx + \frac{n(n-3)}{2}, & n > 2, n \neq 4 \\ 2x^2 + 4, & n = 4. \end{cases}$$

Theorem 8. *$N[C_n; x]$ is stable unless $n = 4$.*

Proof. If $n = 3$, $N[C_3; x] = 3x$. In this case, zero is the only root of $N[C_3; x]$. If $n \neq 3$ or 4, then $N[C_n; x]$ has a single root $x = -(n-3)/2$ which lie in the left half plane. If $n = 4$, the roots of $N[C_4; x]$ are given by $x = \pm\sqrt{2}i$. Hence $N[C_4; x]$ is not stable as it has roots in the right half plane.

Theorem 9. *Let G be a graph with common neighbor polynomial $N[G; x]$ of degree 2. Then the following hold:*

1. *If $N(G, 0) = \phi$ and $N(G, 1) \neq \phi$, then $N[G; x]$ is a stable polynomial.*
2. *If $N(G, 0) \neq \phi$ and $N(G, 1) = \phi$, then $N[G; x]$ is not a stable polynomial.*

Proof. Since $N[G; x]$ is of degree 2, $|N[G, 2]| \neq 0$. We consider the two cases:

1. Let $N(G, 0) = \phi$ and $N(G, 1) \neq \phi$. In this case, the roots of $N[G; x]$ are given by $x = 0$ and $x = -|N(G, 1)|/|N(G, 2)|$. It follows that $N[G; x]$ is stable.
2. Let $N(G, 0) \neq \phi$ and $N(G, 1) = \phi$. Then the roots of $N[G; x]$ are given be $x = \pm(|N(G, 0)|i)/|N(G, 2)|$. Since $N[G; x]$ has roots in the right half plane, $N[G; x]$ is not stable.

This completes the proof.
A Wheel graph $W_n, n > 3$ is obtained by taking the join of the cycle C_{n-1} and K_1.

Corollary 10. *$N[W_n; x]$ is stable unless $n = 5$.*
Proof. We have (see [3]),

$$N[W_n; x] = \begin{cases} \frac{(n-1)(n-4)}{2}x + 2(n-1)x^2, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 4x, & \text{if } n = 5. \end{cases}$$

When $n = 4$, $N[W_4; x] = 6x^2$ which has only one root namely zero. When $n = 5$, the common neighbor roots of W_n are $x = 0, -1 \pm i$. In this case, $N[W_n; x]$ is not stable. When $n \neq 4$ or 5, $N[W_n; x]$ is of degree 2 with $N(W_n, 0) = 0$ and $N(W_n, 1) \neq 0$. So the result follows from Theorem 9.
A shell graph S_n where $n \geq 3$ is obtained from the cycle graph C_n by adding the edges corresponding to the $(n-3)$ concurrent chords of the cycle.The vertex at which all the chords are concurrent is called the apex of the shell.

Corollary 11. *$N[S_n; x]$ is stable.*
Proof. We have, $N[S_n; x] = 2(n-3)x^2 + \left(\binom{n-2}{2} + 3\right)x$.(see [3])
Since $n \geq 3$, $N[S_n; x]$ satisfies the conditions of first part of Theorem 9. Hence it is stable.
A bow graph is a double shell with same apex in which each shell has any order.

Corollary 12. *If B_N is a bow graph where $N > 5$, $N[B_N; x]$ is stable.*
Proof. We have (see [3]), $N[B_N; x] = 2(N-5)x^2 + \left[\frac{N(N-5)}{2} + 10\right]x$.
Since $n > 5$, $N[B_n; x]$ satisfies the conditions of first part of Theorem 9. Hence it is stable.
Here we need the following:

Theorem 13. *(Routh-Hurwitz Criteria [7]) Given a polynomial, $P(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_0$, where the coefficients a_i are real constants, $i = 1, 2, \ldots, n$ define the n Hurwitz matrices using the coefficients a_i of the above polynomial as*

$$H_1 = [a_1] \qquad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \quad \cdots \quad H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: det $H_j > 0$, $j = 1, 2, \ldots, n$.
A Lollipop graph $L_{n,1}$ is obtained by joining K_n to a path of length 1 with a bridge.

Corollary 14. *If $L_{n,1}$ is the Lollipop graph with $n+1$ vertices, then $N[L_{n,1}; x]$ is stable if and only if $n = 4$.*
Proof. We have, $N[L_{n,1}; x] = \binom{n}{2}x^{n-2} + (n-1)x + 1$.(see [3])
When $n = 4$, $N[L_{4,1}; x] = 6x^2 + 3x + 1$. So considering the polynomial $x^2 + \frac{1}{2}x + \frac{1}{6}$, the determinants of Hurwitz matrices are given by, $|H_1| = \frac{1}{2}$ and $|H_2| = \frac{1}{12}$. Since

all the determinants are positive, it follows that $N[L_{4,1};x]$ is stable. When $n > 4$, the result follows from the fact that the determinant of first Hurwitz matrix of $N[L_{n,1};x]$ is zero.
The n- barbell graph $B_{n,1}$ where $n \geq 1$ is a graph obtained by connecting two copies of complete graph K_n by a bridge.

Corollary 15. *If $B_{n,1}$ is the n-Barbell graph with $2n$ vertices, then $N[B_{n,1};x]$ is stable if and only if $n = 4$.*
Proof. We have, $N[B_{n,1};x] = 2\binom{n}{2}x^{n-2} + 2(n-1)x + (n-1)^2 + 1$(see [3]). Now the proof is similar to the proof of Corollary 14.
The bipartite Cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Corollary 16. *If B_n is the bipartite cocktail party graph, then $N[B_n;x]$ is stable if and only if $n = 3$.*
Proof. We have, $N[B_n;x] = 2\binom{n}{2}x^{n-2} + n^2$(see [3]). Now the proof is similar to the proof of Corollary 14.
The Windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common.

Corollary 17. *If $W_n^{(m)}$ is the Windmill graph, then $N[W_n^{(m)};x]$ is stable if and only if $n = 4$.*
Proof. We have, $N[W_n^{(m)};x] = m\binom{n}{2}x^{n-2} + \binom{m}{2}(n-1)^2x$(see [3]). Now the proof is similar to the proof of Corollary 14.

Theorem 18. *Let G be a graph with common neighbor polynomial $N[G;x]$ of degree 2 where $|N(G,i)| > 0$ for $i = 0,1,2$. Then $N[G;x]$ is stable. Moreover, $N[G;x]$ has two negative real roots if $|N(G,1)|^2 \geq 4|N(G,2)||N(G,0)|$ and $N[G;x]$ has two complex roots with negative real parts if $|N(G,1)|^2 < 4|N(G,2)||N(G,0)|$.*
Proof. Since $N[G;x]$ is a polynomial of degree 2, $N[G;x]$ can be represented in the form $N[G;x] = |N(G,2)|x^2 + |N(G,1)|x + |N(G,0)|$. The Hurwitz matrices of $N[G;x]$ are given by $H_1 = \begin{bmatrix} \frac{|N(G,1)|}{|N(G,2)|} \end{bmatrix}$ and $H_2 = \begin{bmatrix} \frac{|N(G,1)|}{|N(G,2)|} & 1 \\ 0 & \frac{|N(G,0)|}{|N(G,2)|} \end{bmatrix}$. Since $|N(G,i)| > 0$ for $i = 0,1,2$; it follows that $\det H_1 > 0$ and $\det H_2 > 0$. Hence by Theorem 9, $N[G;x]$ is stable so that all the roots of $N[G;x]$ lie in the left half plane. Moreover, the discriminant of $N[G;x]$ is given by $\Delta = |N(G,1)|^2 - 4|N(G,2)||N(G,0)|$. Then $N[G;x]$ has 2 real roots if $\Delta \geq 0$ and has two complex roots if $\Delta < 0$. Hence the result follows.
A helm, $H_n, n > 3$ is obtained from a wheel graph W_n by adding pendent edges to every vertices on the wheel rim.

Corollary 19. *If H_n, $n > 3$ is a helm graph with $2n-1$ vertices, then $N[H_n; x]$ is stable. Moreover, $N[H_n; x]$ has two negative real distinct zeros if $n > 40$ and has two complex zeros with negative real parts if $n \leq 40$.*
Proof. We have, (see [3]),

$$N[H_n; x] = \begin{cases} 2(n-1)x^2 + \frac{(n-1)(n+2)}{2}x + \frac{(n-1)(3n-8)}{2}, & \text{if } n \neq 5 \\ 2x^3 + 4x^2 + 16x + 14, & \text{if } n = 5. \end{cases}$$

We consider two cases.

1. Let $n = 5$. In this case, $N[H_5; x] = 2x^3 + 4x^2 + 16x + 14$. So considering the equation, $x^3+2x^2+8x+7 = 0$, the values of the determinants of corresponding Hurwitz matrices are given by, $|H_1| = \big|2\big| = 2$, $|H_2| = 9$ and $|H_3| = 63$. Since determinants of all Hurwitz matrices are positive, by Theorem 13, $N[H_5; x]$ is stable.

2. Let $n \neq 5$. Since $n > 3$, $N[H_n; x]$ satisfies the conditions of Theorem 18 and hence it is stable. Moreover, the discriminant of $N[H_n; x]$ is given by,

$$\Delta = \tfrac{(n-1)^2}{4}\Big[n^2 - 44n + 132\Big]$$

   Now $n > 3$ and $\Delta > 0$ implies that $n > 40$. Hence the result follows from Theorem 18.

A web graph $WB_n, n > 3$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle.

Corollary 20. *If WB_n is a web graph with $3(n-1)$ vertices, then $N[WB_n; x]$ is stable. Moreover, $N[WB_n; x]$ have two real distinct roots if $n > 241$ and have two complex roots with negative real parts if $n \leq 241$.*
Proof. We have, (see [3]),

$$N[WB_n; x] = \begin{cases} 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x + (n-1)(4n-10), & \text{if } n \neq 5 \\ 2x^3 + 14x^2 + 20x + 42, & \text{if } n = 5. \end{cases}$$

We consider two cases.

1. Let $n = 5$. In this case, $N[WB_n; x] = 2x^3 + 14x^2 + 20x + 42$. So considering the equation, $x^3 + 7x^2 + 10x + 21 = 0$, the values of the determinants of corresponding Hurwitz matrices are given by, $|H_1| = 7$, $|H_2| = 49$ and $|H_3| = 1029$. Since determinants of all Hurwitz matrices are positive, by Theorem 9, $N[WB_5; x]$ is stable.

2. Let $n \neq 5$. Since $n > 3$, $N[WB_n; x]$ satisfies the conditions of Theorem 13 and hence it is stable. Moreover, the discriminant of $N[WB_n; x]$ is given by,

$$\Delta = \tfrac{(n-1)^2}{4}\left[n^2 - 244n + 676\right]$$

Now $n > 3$ and $\Delta > 0$ implies that $n > 241$. Hence the result follows from Theorem 18.

A butterfly graph is a bow graph along with exactly two pendent edges at the apex.

Corollary 21. *If WF is a butterfly graph with $N > 7$ vertices, then $N[WF; x]$ is stable.*

Proof. We have, (see [3]),

$$N[BF; x] = 2(N-7)x^2 + \left[\frac{N(N-5)}{2} + 12\right]x + 2.$$

Since $N > 7$, $N[BF; x]$ satisfies the conditions of Theorem 18 and hence it is stable.

Theorem 22. *If $K_{m,n}$ is a complete bipartite graph with $m + n$ vertices, where $n > m$ then $N[K_{m,n}; x]$ is stable if and only if $n = 2$ and $m = 1$.*

Proof. We have, $N[K_{m,n}; x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn$ where $n, m \geq 2$.(see [3])
Clearly, the condition is sufficient since, $N[K_{1,2}] = x + 2$ which has root $x = -2$ lying in the left half plane.
To prove the necessary part, we consider two cases.

Case(i) Let $n - m = 1$

In this case $N[K_{m,n}; x]$ can be expressed as,

$N[K_{m,n}; x] = \binom{m}{2}[x^n + a_1x^{n-1} + a_n]$, where $a_1 = \frac{\binom{n}{2}}{\binom{m}{2}}$ and $a_n = \frac{mn}{\binom{m}{2}}$.

Now, considering the polynomial $x^n + a_1x^{n-1} + a_n$, the determinants of first two Hurwitz matrices are given by $|H_1| = a_1 > 0$ and $|H_2| = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} =$

$$\begin{cases} a_1a_2 > 0 \text{ if } & n = 2 \\ -a_3 < 0 \text{ if } & n = 3 \\ \quad 0 \quad\ \text{ if } & n \geq 4 \end{cases}$$

Hence determinants of all the Hurwitz matrices are positive only if $n = 2$. Then by the assumption of Case(i), $m = 1$.

Case(ii) Let $n - m > 1$

Then the determinant of first Hurwitz matrix of $N[K_{m,n}; x]$ itself is zero for all m, n.

Hence from Theorem 13, it follows that $N[K_{m,n}; x]$ is stable only if $n = 2$ and $m = 1$.

References

[1] Gajender, Gaurav and Himanshu Sharma, Hurwitz polynomial, International journal of innovative research in technology, Vol.1(7), 2014.

[2] Harary F., Graph Theory, Adison-Wesley,1969.

[3] M. Shikhi and V. Anilkumar, Common neighbor polynomial of graphs, Far East Journal of mathematical sciences,Vol. 102, No. 6, 2017, Pages 1201-1221.

[4] M. Shikhi and V. Anilkumar, Common neighbor polynomial of graph operations, Far East Journal of mathematical sciences(accepted).

[5] M. Shikhi and V. Anilkumar, CNP-equivalent Classes of Graphs, South East Asian Journal of Mathematics and Mathematical Sciences, Vol.13, No.2, 2017, pages 75-84.

[6] N.K. Vishnoi, Zeros of polynomials and their applications to theory: A primer, Preprint, Microsoft Research, Bangalore, India, 2013.

[7] P. A. Fuhrmann, A polynomial approach to linear algebra, Springer-Berlin Heidelberg, New York, 1996.